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## Singular Soliton Operators and Indefinite Metrics

**Abstract.** *The singular real second order 1D Schrödinger operators are considered here with such potentials that all local solutions near singularities to the eigenvalue problem are meromorphic for all values of the spectral parameter. All algebro-geometrical or "singular finite-gap" potentials as well as singular solitons satisfy to this condition. A Spectral Theory is constructed here for the periodic and rapidly decreasing cases in the special classes of functions with singularities and indefinite inner product. It has a finite number of negative squares if the unimodular Bloch multipliers are fixed in the periodic case, and in the rapidly decreasing case. The time dynamics provided by the KdV hierarchy preserves this number. The right analog of Fourier Transform for the Riemann Surfaces preserving remarkable multiplicative properties of the ordinary (i.e. genus zero) Fourier Transform based on the standard exponential basis, leads to such operators as it was shown in our previous works.*

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# The Main Constructions and Results

## The Fourier Transform and Riemann Surfaces

Consider real  $\infty$ -smooth potentials  $u(x)$  meromorphic in a small complex area near the point  $x_j \in \mathbb{R}$  as in [1, 2]. Following statement can be easily proved:

**Statement 1.** *All solutions to the Sturm-Liouville equations (for all  $\lambda$ )*

$$L\Psi = -\Psi'' + u(x)\Psi = \lambda\Psi$$

*are meromorphic in a small area near the point  $x_j$  if and only if*

$$u(x) = n_j(n_j + 1)/(x - x_j)^2 + \sum_{k=0}^{n_j-1} u_{jk}(x - x_j)^{2k} + O((x - x_j)^{2n_j})$$

*for some  $n_j \in \mathbb{Z}$ . There exists a basis of solutions near  $x_j$  such that for  $y = x - x_j$  and all  $\lambda$*

$$\psi_{1j} = 1/y^{n_j} + a_1(\lambda)/y^{n_j-2} + a_2(\lambda)/y^{n_j-4} + \dots + a_{n_j}(\lambda)/y^{-n_j} + O(y^{n_j+1}) \quad (1)$$

$$\psi_{2j} = y^{n_j+1} + \dots$$

**Statement 2.** *All algebrogeometric (AG) potentials satisfy to the conditions of the Statement 1 (there exists a differential operator  $A$  of an odd order such that  $[L, A] = 0$  for the AG case).*

We call such potentials “the singular finite-gap potentials”  $u(x)$  if they are periodic in  $x$ :  $u(x + T) = u(x)$ . We call them “the singular solitons” if  $u(x) \rightarrow 0$ ,  $|x| \rightarrow \infty$ .

The simplest important examples famous in the classical literature are “the singular solitons”:

$$u(x) = \frac{n(n+1)}{x^2}, \quad u(x) = \frac{n(n+1)k^2}{\sinh^2(kx)}$$

and “the Lamé potentials” (degenerate and non-degenerate)

$$u(x) = \frac{n(n+1)k^2}{\sin^2(kx)}, \quad u(x) = n(n+1)\wp(x)$$

The “Dirichlet Problem” for the real Lamé potentials at the interval  $[0, T]$  with real period  $T = 2\omega$  and imaginary period  $T' = 2\omega'$  was studied by Hermit. No good spectral theory for the Lamé operators on the whole line  $\mathbb{R}$  in the Hilbert space  $\mathcal{L}_2(\mathbb{R})$  was known. **We are going to construct a spectral theory for the operator  $L$  of this type in the space of functions on the real line with Indefinite Inner Product.**

**Fix the set  $X$  of points  $x_j \in \mathbb{R}$ ,  $j = 1, \dots, N$  and numbers  $n_j \in \mathbb{Z}_+$ .** This set should be finite for the rapidly decreasing potentials  $u(x) = O(1/x^2)$ ,  $|x| \rightarrow \infty$ . It should be finite at each period  $(x, x + T)$  for the periodic case. For the periodic case we fix also a unitary Bloch multiplier  $\varkappa$ ,  $|\varkappa| = 1$ , where

$$\Psi(x + T) = \varkappa \Psi(x).$$

We choose class of functions  $\mathcal{F}_X^0$ ,  $\infty$ -smooth outside the points  $x_j$  (and their periodic shifts), such that near  $(x_j)$  we have

$$\begin{aligned} \Psi(y) + (-1)^{n_j+1} \Psi(-y) &= O(y^{n_j+1}), \\ y = x - x_j, \quad j &= 1, \dots, N. \end{aligned} \tag{2}$$

The whole space of functions  $\mathcal{F}_X \ni \mathcal{F}_X^0$  consists of functions  $\Psi$  with ”principal part”  $\Phi_j$  at the points  $x_j \in X$

$$\begin{aligned} \Phi_j(y) &= \sum_{k=0}^{n_j} a_{jk} / y^{n_j-2k}, \quad y = x - x_j. \\ \Psi &= \Phi_j + O(y^{n_j+1}), \end{aligned} \tag{3}$$

so the difference  $\Psi - \Phi_j$  belongs locally to the space  $\mathcal{F}_X^0$  near  $x_j$  for all  $j = 1, \dots, N$ . Even more, this difference has the order  $O(y^{n_j+1})$  near  $x_j$ .

The standard inner product

$$\langle \Psi_1, \Psi_2 \rangle = \int_0^T \Psi_1(x) \overline{\Psi_2(x)} dx, \quad u(x + T) = u(x),$$

or

$$\langle \Psi_1, \Psi_2 \rangle = \int_{-\infty}^{\infty} \Psi_1(x) \overline{\Psi_2(x)} dx, \quad u(x) \rightarrow 0, \quad |x| \rightarrow \infty$$

can be extended to the class  $\mathcal{F}_X$  by writing

$$\langle \Psi_1, \Psi_2 \rangle = \int_0^T \Psi_1(x) \overline{\Psi_2(\bar{x})} dx, \quad u(x+T) = u(x),$$

(4)

or

$$\langle \Psi_1, \Psi_2 \rangle = \int_{-\infty}^{\infty} \Psi_1(x) \overline{\Psi_2(\bar{x})} dx, \quad u(x) \rightarrow 0, \quad |x| \rightarrow \infty$$

and avoiding the singular points at the complex domain. Our requirements imply that the product  $\Psi_1(x) \overline{\Psi_2(\bar{x})}$  is meromorphic and has residues near the singularities equal to zero. So our inner product is well-defined (but became indefinite).

**Statement 3.** *The inner products (4) are well-defined with this regularization. They are indefinite with exactly  $l_X = \sum_{j=1}^N l_{n_j}$  negative squares for each  $\varkappa \in S^1$  where*

$$l_{n_j} = \left\lfloor \frac{n_j + 1}{2} \right\rfloor$$

It is positive in the subspace  $\mathcal{F}_X^0 \in \mathcal{F}_X$  by definition. Every coefficient  $a_{jk}$ ,  $k = 0, \dots, l_{n_j} - 1$  (i.e. corresponding to the negative powers of  $y$ ) gives exactly one negative square. The positive powers of  $y$  do not destroy positivity of inner product.

A detailed proof of Statement 3 is presented in the Appendix 2.

Let  $\Gamma$  be a real hyperelliptic Riemann Surface  $w^2 = R_{2g+1}(z)$  of the Bloch-Floquet function  $\Psi_{\pm}(x, z)$  in the “finite-gap” periodic case. It has exactly two antiholomorphic involutions  $\tau_{\pm}$  where  $z \rightarrow \bar{z}$ . We choose the one such that an infinite cycle (i.e. the spectral zone where  $z \in \mathbb{R}$  and  $z \rightarrow +\infty$ ) belongs to the fix point set. The poles of  $\Psi$  are  $x$ -independent and form a divisor  $\mathcal{D}$  consisting of  $g$  points where  $g$  is a genus. The canonical contour  $p_0 \in \Gamma$  containing all points with unimodular multipliers  $|\varkappa| = 1$  is defined (see[1]). We assume in the decomposition theorem below that it is nonsingular. It is invariant under the action of antiinvolution. The infinite component of canonical contour contains an infinite point  $\infty \in \Gamma$ . The antiinvolution is identical on the real part of that component. Fixing  $\varkappa$  we choose a countable number of points  $z_q = (\lambda_q(\varkappa), \pm)$  in the canonical contour. They correspond

to the functions  $\Psi_q = \Psi(x, z_q)$ . Except of finite number, all these points belong to the infinite component. Our Spectral Transform maps the space of  $C^\infty$ -functions in the canonical contour (properly decreasing at infinity) into the space of functions  $\mathcal{F}_X$  in the real line  $R$ . It preserves an indefinite metric as it was proved in [1]. **In the present work we describe the image of this Transform.** It is the space  $\mathcal{F}_X$ . The case of smooth real periodic operators (i.e. the set  $X$  is empty) corresponds to such cases that the Riemann surface has all branching points real, and the divisor  $\mathcal{D}$  such that every finite gap cycle contains exactly one simple pole (see [3]. The union of all gaps is exactly equal to the fixpoint set of the second antiinvolution of the Riemann Surface  $\Gamma$ . The case of Fourier Transform corresponds to the case where all branching points are real and  $\mathcal{D} = g \times \infty$ . It has the best possible multiplicative properties of basis similar to the properties of the standard exponential basis in the ordinary Fourier Transform (i.e. the genus zero surface is  $w^2 = z$ , and canonical contour is an infinite cycle over the line  $z \geq 0, z \in \mathbb{R}$ ). In the Fourier Transform case every singular point  $x_j$  is such that  $n_j$  is equal to the genus, and there is only one singular point at the period. There exists an operator  $R$  with coefficients dependent on  $x, y, x + y$  but independent on the point  $P$  of Riemann surface  $R = \partial_x^g + a_1 \partial_x^{g-1} + \dots$  such that for the Baker-Akhiezer functions we have

$$\Psi(x, P)\Psi(y, P) = R\Psi(x + y, P)$$

We have  $a_1 = -(\zeta(x + \zeta(y) - \zeta(x + y)))$  for  $g = 1$  and Lamé potential (the Hermit case). Easy to describe coefficients also for all Riemann surfaces.

Discrete version of Fourier Transform with good multiplicative properties was developed by Krichever and Novikov in the series of works in the late 1980s to develop the operator construction of the bosonic (Polyakov type) string theory for all diagrams which are the Riemann surfaces of all genres (see in the book [6]).

**Theorem 1.** *Every function  $f \in \mathcal{F}_X$  such that  $f(x + T) = \varkappa f(x)$  can be uniquely presented in the form*

$$f = \sum_q c_q \Psi_q, \quad \lambda_q = \lambda_q(\varkappa), \quad c_q = \langle f, \Psi_q \rangle / \langle \Psi_q, \Psi_q \rangle.$$

where  $L\Psi_q = \lambda_q \Psi_q$ ,  $\Psi_q = \Psi(x, z_q)$ ,  $z_q = (\lambda_q(\varkappa), +)$  or  $z_q = (\lambda_q(\varkappa), -)$ , and  $u(x)$  is a real periodic singular finite-gap potential. This series for the

singular parts is convergent (more rapidly than any power). In some neighborhood of the points  $x_j$  the series  $\sum_q (\Psi_q - \Phi_{qj})c_q$  converges to the corresponding differences  $(\Psi - \Phi_j)$  with all derivatives, near every point  $x_j$ . Here  $\Phi_{qj} = \sum_{k=0}^{n_j-1} a_{(q)jk}/y^{n_j-2k}$ ,  $y = x - x_j$ , which are the principal parts of the eigenfunctions  $\Psi_q$ , at the points  $x_j \in X$ , and  $\Phi_j$  is the principal part of  $f \in \mathcal{F}_X$  as it was defined above in Formula 3.

**Theorem 1'.** Consider a rapidly decreasing potential  $u(x)$ . For every function  $f \in \mathcal{F}_X$  decreasing rapidly enough at  $|x| \rightarrow \infty$ , we have the following presentation

$$f = \int_{k \in \mathbb{R}} c_k \Psi_k(x) dk + \sum_m d_m \Psi_m,$$

$$L\Psi_k = k^2 \Psi_k, \quad L\Psi_m = \lambda_m \Psi_m,$$

Here  $u(x) = O(1/x^2)$  at  $|x| \rightarrow \infty$ , and we assume that  $u(x)$  is a singular multisoliton potential.

Such results for the standard positive Hilbert spaces and regular self-adjoint 1D stationary Schrödinger operators were known many years at folklore level (see the formulas and quotations in the article [3]). For more complicated situation of non-stationary 1D Schrödinger operators and stationary 2D Schrödinger operators specific formulas and decomposition theorems on Riemann surfaces were defined in the original works [4], [5]. We use these technique for the extension to our indefinite case.

**Our program is to extent in the next works these results to the whole class of periodic and rapidly decreasing infinite-gap real periodic potentials with singularities of the type described above.**

By the way, in the work [7] the “scattering data” were constructed for the case  $u(x) = O(1/x^2)$  at  $|x| \rightarrow \infty$ , all  $n_j = 1$ . Indefinite metric, spectral theory and decomposition of functions were not discussed in this work.

An interesting application of our theory is connected with the following problem: consider the KdV solutions  $u_t = 6uu_x - u_{xxx}$  such, that  $u(x, 0) = n(n+1)/x^2$ . It is well-known, that we can write these solutions in the form

$$u(x, t) = 2 \sum_{q=1}^{n(n+1)/2} \frac{1}{(x - x_q(t))^2},$$

It is easy to see, that  $x_q = a_q t^{1/3}$ . How many of  $x_j$  are real?

**In our paper we prove, that exactly  $l_n = \lfloor \frac{n+1}{2} \rfloor$  poles remain real. This number is exactly equal to the number of coefficients  $a_{kj}$  in every singular point  $x_j$  with  $n_j = n$ .**

**Remark** The following transformations preserve the set  $\{a_q\}$ :  $a_q \rightarrow \bar{a}_q$ ,  $a_q \rightarrow \xi a_q$ ,  $\xi^3 = 1$ .

**Proof.** It is clear, that this problem is equivalent to the following one: consider the KdV solutions  $u_t = 6uu_x - u_{xxx}$  such, that  $u(x, 0) = n(n+1)\wp(x)$ , where  $\wp$  is the Weierstrass function, associated with a real rectangular lattice. How many poles on the period are real for  $t > 0$ ,  $t \ll 1$ ?

Assume, that a generic unitary Bloch multiplier  $\varkappa_0$  is fixed. From Appendix 2 it follows, that the space  $\mathcal{F}_X$  has exactly  $l_n = \lfloor \frac{n+1}{2} \rfloor$  negative squares for  $t = 0$ . From Theorem 1 proved in Appendix 1 it follows, that any collection of singularities can be approximated by the image of the Fourier map, therefore the number of negative squares is equal to the number of points  $\gamma$  on the canonical contour such, that  $\exp(ip(\gamma)T) = \varkappa_0$  and  $d\mu(\gamma)/dp(\gamma) < 0$ , where  $d\mu$  is the spectral measure in the decomposition formula

$$d\mu = \frac{(\lambda(\gamma) - \lambda_1) \dots (\lambda(\gamma) - \lambda_g)}{2\sqrt{(\lambda(\gamma) - E_0) \dots (\lambda(\gamma) - E_{2g})}} d\lambda(\gamma).$$

This number does not depend on  $t$ , therefore the number of negative squares in  $\mathcal{F}_X$  also does not depend on  $t$ . For small  $t > 0$  all singularities are simple, therefore the number of negative squares coincide with the number of real singular points on the period. This completes the proof.

Let us point out, that in [1] we already have proved that  $l'_n \geq l_n$  where  $l'_n$  is the number of real  $a_q$ . Here  $l_n$  is equal to the number of negative squares in the inner product above for this specific case. This quantity is time-invariant. Naive understanding of the opposite inequality  $l'_n \leq l_n$  is the following: numerical calculations show, that the points  $x_q(t)$  for small  $t > 0$ ,  $t \in \mathbb{R}$  are localized approximately in the points of the equilateral triangle (see Fig 1).

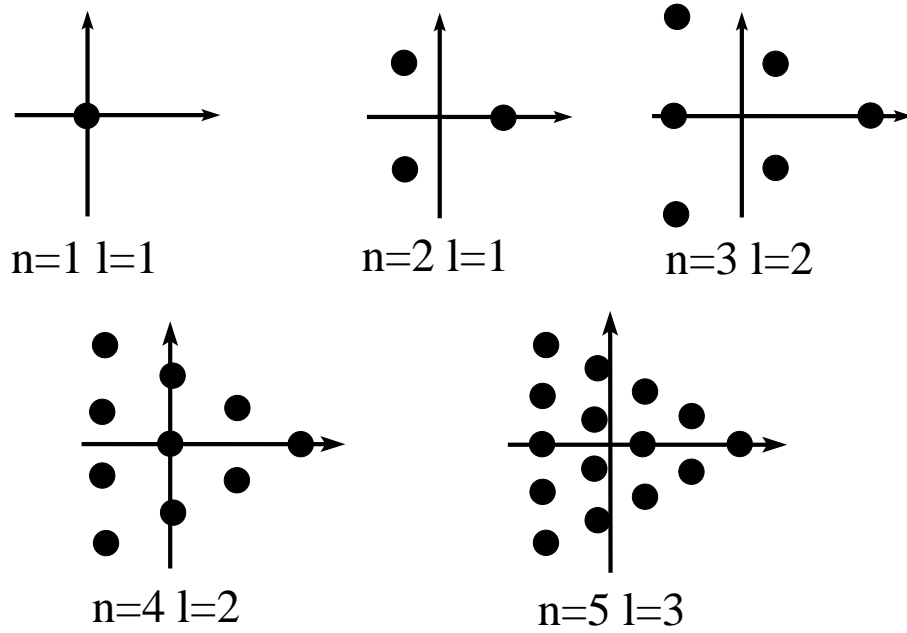


Fig 1.

The poles  $a_q$  for different values of  $n$ .

For the case of the ideal equilateral triangle we obviously have  $l'_n = l_n = \left\lceil \frac{n+1}{2} \right\rceil$ .

However, in fact, it is slightly perturbed. So we should have  $l'_n \leq l_n$  if perturbation is really small. But the symmetry  $a_q \rightarrow \bar{a}_q$  keeps all real points on the real line. So we are done with the really small perturbations of the equilateral triangle. But our perturbation is only numerically small, not theoretically. So this argument is non-rigorous.

**Remark.** The positions of these zeroes were studied numerically and analytically in [8]. The problem of calculation of the number of real zeroes was not discussed in [8], and it is not clear, if it is possible to obtain a rigorous proof of our result using the estimates from this paper. The first rigorous proof of the inequality  $l'_n \leq l_n$  was completed with the help of student A.Fetisov.

A non-standard example we obtain for the case of elliptic function  $u(x) = 2\wp(x)$  corresponding to the rhombic lattice (see Fig. 2a. Fig. 2b). The canonical contour is connected in this case. It has two singular points. The antiinvolution is not identity at the contour in this case, so there are no self-adjoint real problems on the real line for such Riemann surface. The inner product is always indefinite. The projection of contour on the plane



of spectral parameter contains complex part, so the spectrum of operator is complex for such real singular finite-gap potential.

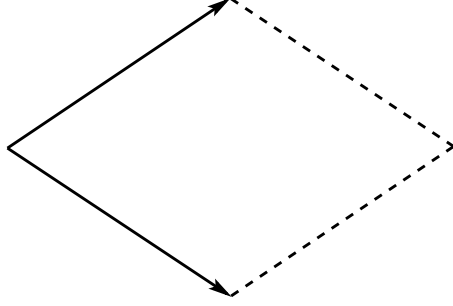


Fig 2a  
The rhombic lattice

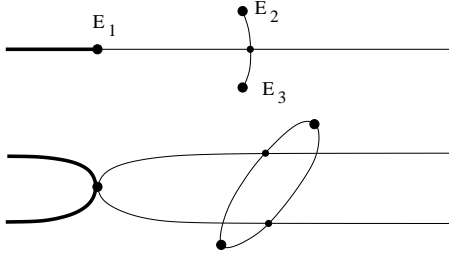


Fig 2b  
The contour  $|\varkappa| = 1$  is singular

The classical Lamé problems do not lead to this case, so it never was considered.

## 1 Appendix. Proof of Theorem 1

In the Appendix we prove Theorem 1. We construct also an analog of the continuous Fourier decomposition by the eigenfunctions of a singular finite-gap operator with a **periodic** potential.

The proof consists of two steps.

1. We reduce the decomposition problem for real singular potential to the decomposition problem for regular complex potentials.
2. We construct eigenfunction expansion for regular complex potentials.

### 1.1 Notations

Let us recall the basis definitions. Let the spectral curve  $\Gamma$  be defined by:

$$\mu^2 = (\lambda - E_0) \dots (\lambda - E_{2g}) = R(\lambda).$$

Denote our divisor by  $\mathcal{D} = \gamma_1 + \dots + \gamma_g$ .

In Appendix we use the following notations:  $\lambda(\gamma)$  denotes the projection of the point  $\gamma \in \Gamma$  to the  $\lambda$ -plane. Equivalently either  $\gamma = (\lambda(\gamma), +)$  or  $\gamma = (\lambda(\gamma), -)$ .

Let  $\lambda_1 = \lambda(\gamma_1), \dots, \lambda_g = \lambda(\gamma_g)$ .

The quasimomentum differential  $dp$  is uniquely defined by the following properties:

1.  $dp$  is holomorphic in  $\Gamma$  outside the point  $\lambda = \infty$ .
- 2.

$$dp = dk \left( 1 + O\left(\frac{1}{k^2}\right) \right), \quad k^2 = \lambda$$

near the point  $\lambda = \infty$ .

3. Integrals over all basic cycles are purely real

$$\operatorname{Im} \oint_c dp = 0 \tag{5}$$

for any closed contour  $c \subset \Gamma$

The quasimomentum function  $p(\gamma)$  is the primitive of  $dp$ , and it is always multivalued. We assume, that

$$p(\gamma) = k + O\left(\frac{1}{k}\right).$$

From (5) it follows, that the imaginary part of the quasimomentum function  $\operatorname{Im} p(\gamma)$  is well-defined.

We assume, that our potential  $u(x)$  is periodic with the period  $T$ . It implies, that

$$\exp(ip(\gamma)T)$$

is a single-valued function in  $\Gamma$ .

As above, we denote the Bloch function by  $\Psi(\gamma, x)$ , and  $\sigma$  denotes the holomorphic involution, interchanging the sheets of the surface  $\Gamma$ :

$$\sigma : (\lambda, +) \rightarrow (\lambda, -)$$

Assume, that function  $f(x)$  has finite support and  $f \in \mathcal{F}_X$ .

Let us define the continuous Fourier transform for  $f(x)$  by:

$$\hat{f}(\gamma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi^*(y, \gamma) f(y) dy \quad (6)$$

where we use the rule of avoiding the singularities by going into the complex domain.

**Theorem 1”.** *Assume, that:*

1. *The spectral curve  $\Gamma$  is regular (has no multiple points),*
2. *The contour  $\text{Im } p(\gamma) = 0$  is regular, i.e.  $dp(\gamma) \neq 0$  everywhere at this contour.*

*Then we have the following reconstruction formula:*

$$f(x) = \oint_{\text{Im } p(\gamma)=0} \hat{f}(\gamma) \Psi(x, \gamma) \frac{(\lambda(\gamma) - \lambda_1) \dots (\lambda(\gamma) - \lambda_g)}{2\sqrt{(\lambda(\gamma) - E_0) \dots (\lambda(\gamma) - E_{2g})}} d\lambda(\gamma). \quad (7)$$

*For any regular  $x$  The integrand in 7 decays for  $\gamma \rightarrow \infty$  faster, than any degree of  $\lambda$ .*

## 1.2 Reduction to a smooth potential

Let us show, that applying a series of Crum transformations we can reduce the decomposition with respect to singular potential to the decomposition with respect to complex potentials.

**Lemma 1.** *Let  $n = n_{\max}$  denotes the maximal order of singularity  $n_{\max} = \max_j n_j$ . Then applying a series of  $n$  properly chosen Darboux-Crum transformation one can obtain a regular **complex** potential.*

**Proof.**

Consider the image of all divisor trajectories in  $\Gamma$ . Potential  $u(x)$  is periodic, therefore they form a compact set. Let  $q_1 \in \Gamma$  be a point outside this set such, that  $\text{Im } p(q_1) \neq 0$ ,  $l_1 = \lambda(q_1)$ .

Let  $\psi_1(x) = \Psi(q_1, x)$  be the corresponding Baker-Akhiezer function.

$$L\psi_1(x) = l_1\psi_1(x).$$

Then

1.  $\psi_1(x) = \frac{1}{(x-x_j)^{n_j}}(a_0^{(j)} + o(1))$ ,  $a_0^{(j)} \neq 0$  at all singular points.
2.  $\psi_1(x) \neq 0$  for all  $x \in \mathbb{R}$ ,  $x \neq x_j$ .

Let

$$Q_1 = \left( \partial_x - \frac{\psi_x}{\psi} \right), \quad Q_1^* = \left( -\partial_x - \frac{\psi_x}{\psi} \right).$$

We have

$$L - l_1 = Q_1^* Q_1, \quad L_1 - l_1 = Q_1 Q_1^*,$$

where  $L_1$  denotes the Darboux-Crum transformed operator,

$$u^{(1)}(x) = u(x) - 2\partial_x^2 \log(\psi_1)$$

$$\Psi^{(1)}(x, \gamma) = \frac{1}{\lambda - l_1} Q_1 \Psi(x, \gamma).$$

We see, that this transformation reduced the orders of all singularities to 1 and generates no new singular points. By repeating this procedure  $n$  times we come to a smooth potential  $u^{(n)}(x)$ .

Denote the corresponding operators  $L = L_0, L_1, \dots, L_n$ ,

$$L_n = -\partial_x^2 + u^{(n)}(x), \quad L_k - l_k = Q_k Q_k^*, \quad L_k - l_{k+1} = Q_{k+1}^* Q_{k+1},$$

$$Q_k Q_k^* = Q_{k+1}^* Q_{k+1} + l_{k+1} - l_k$$

This procedure generates Bloch functions with a slightly non-standard normalization. To obtain the standard Baker-Akhiezer function, it is necessary to change the normalization of  $\Psi(x, k)$ .

Denote by  $\gamma_1(x), \dots, \gamma_g(x)$  the divisor of zeroes of  $\Psi(x, \gamma)$ .

Let  $x_0$  be one of the singular points with the highest order singularity. It means, that for  $x = x_0$  exactly  $n$  points of the divisor  $\gamma_1(x_0), \dots, \gamma_g(x_0)$  are located at the point  $\lambda = \infty$ . Denote the remaining points by  $\gamma_1(x_0), \dots, \gamma_{g-n}(x_0)$ .

Let  $\tilde{\Psi}(x, \gamma)$  be the Baker-Akhiezer function with  $g-n$  simple poles  $\gamma_1(x_0), \dots, \gamma_{g-n}(x_0)$  on the finite part of  $\Gamma$  and asymptotics

$$\tilde{\Psi}(x, \gamma) = e^{ik(x-x_0)}((-ik)^n + O(k^{n-1})), \quad k^2 = \lambda, \quad \lambda \rightarrow \infty. \quad (8)$$

Then

$$\tilde{\Psi}^{(n)}(x, \gamma) = Q_n Q_{n-1} \dots Q_1 \tilde{\Psi}(x, \gamma) \quad (9)$$

is the Baker-Akhiezer function for the smooth operator  $L_n$  with the divisor of poles  $\gamma_1(x_0), \dots, \gamma_{g-n}(x_0), \sigma q_1, \dots, \sigma q_n$  and essential singularity

$$\tilde{\Psi}^{(n)}(x, \gamma) = e^{ik(x-x_0)}(1 + o(1)), \quad \lambda \rightarrow \infty. \quad (10)$$

**Lemma 2.** *Operators  $Q_j, Q_j^*$  map Bloch functions to the Bloch functions with the same multiplier.*

Consider the following operator:

$$M = Q_n \cdot \dots \cdot Q_1 \cdot Q_1^* \cdot \dots \cdot Q_n^*$$

**Lemma 3.** *We have the following formula*

$$M = (L_n - l_1)(L_n - l_2) \dots (L_n - l_n)$$

The proof is straightforward:

$$\begin{aligned} Q_n \dots Q_4 Q_3 Q_2 Q_1 Q_1^* Q_2^* Q_3^* Q_4^* \dots Q_n^* &= Q_n \dots Q_4 Q_3 Q_2 (Q_2^* Q_2 + l_2 - l_1) Q_2^* Q_3^* Q_4^* \dots Q_n^* = \\ &= Q_n \dots Q_4 Q_3 (Q_2 Q_2^* + l_2 - l_1) Q_2 Q_2^* Q_3^* Q_4^* \dots Q_n^* = \\ &= Q_n \dots Q_4 Q_3 (Q_3^* Q_3 + l_3 - l_1) (Q_3^* Q_3 + l_3 - l_2) Q_3^* Q_4^* \dots Q_n^* = \\ &= Q_n \dots Q_4 (Q_3 Q_3^* + l_3 - l_1) (Q_3 Q_3^* + l_3 - l_2) Q_3 Q_3^* Q_4^* \dots Q_n^* = \\ &= Q_n \dots Q_4 (Q_4^* Q_4 + l_4 - l_1) (Q_4^* Q_4 + l_4 - l_2) (Q_4^* Q_4 + l_4 - l_3) Q_4^* \dots Q_n^* = \\ &= Q_n \dots (Q_4 Q_4^* + l_4 - l_1) (Q_4 Q_4^* + l_4 - l_2) (Q_4 Q_4^* + l_4 - l_3) Q_4 Q_4^* \dots Q_n^* = \\ &\dots \\ &= (Q_n Q_n^* + l_n - l_1) (Q_n Q_n^* + l_n - l_2) \dots (Q_n Q_n^* + l_n - l_{n-1}) Q_n Q_n^* = \\ &= (L_n - l_1) (L_n - l_2) \dots (L_n - l_{n-1}) (L_n - l_n) \end{aligned}$$

**Corollary 1.** *Operator  $M$  a differential operator with smooth coefficients.*

**Remark 1.** *From the definition of  $Q_k$  it follows immediately, that*

$$Q_1^* \dots Q_n^* \cdot Q_n \cdot \dots \cdot Q_1 \Psi(x, \gamma) = (\lambda(\gamma) - l_n) \dots (\lambda(\gamma) - l_1) \Psi(x, \gamma) \quad (11)$$

**Lemma 4.** *Let:*

$$f^{(n)}(x) = Q_n \cdot Q_{n-1} \cdot \dots \cdot Q_1 f(x),$$

*where  $f \in \mathcal{F}_X$ . Then  $f^{(n)}(x)$  is a complex smooth periodic potential.*

The proof is straightforward: each operator  $Q_k$  reduces the order  $n_j$  of singularity at the point  $x_j$  by 1.

**Lemma 5.** *Assume, that the function  $f^{(n)}(x)$  admits the Laurent-Fourier decomposition by the Bloch functions for  $L_n$ :*

$$of^{(n)}(x) = \sum_j c_j \tilde{\Psi}^{(n)}(\kappa_j, x),$$

Then the function  $M^{-1}f^{(n)}(x)$  is well-defined, and we have

$$M^{-1}f^{(n)}(x) = \sum_j \frac{c_j}{(\lambda(\kappa_j) - l_1) \dots (\lambda(\kappa_j) - l_n)} \tilde{\Psi}^{(n)}(\kappa_j, x).$$

For all  $\kappa_j$   $\text{Im } p(\kappa_j) = 0$ , therefore we have no zeroes in the denominators.

We also have:

$$f^{(n)}(x) = M(M^{-1}f_n(x)) = Q_n \cdot \dots \cdot Q_1 \cdot Q_1^* \cdot \dots \cdot Q_n^* \cdot (M^{-1}f^{(n)}(x)),$$

therefore

$$f(x) = Q_1^* \cdot \dots \cdot Q_n^* \cdot (M^{-1}f^{(n)}(x)). \quad (12)$$

The results of this section can be summarized in the following way. To decompose the given function  $f(x)$  we:

1. By applying  $n$  properly chosen Darboux-Crum transformations we obtain a smooth functions  $f^{(n)}(x) = Q_n \cdot Q_{n-1} \cdot \dots \cdot Q_1 f(x)$ .
2. We expand the smooth function  $f^{(n)}(x)$  by the eigenfunctions of **smooth complex finite-gap** operator  $L_n$ .
3. Taking into account, that all points  $q_n$  are located outside the contour  $\text{Im}(p) = 0$ , we construct eigenfunctions expansion for  $(M^{-1}f_n(x))$ .
4. Applying formula (12) we obtain an eigenfunction decomposition of  $f(x)$ . At this stage we use the following property of the Darboux-Crum transformation: operator  $Q_1^* \cdot \dots \cdot Q_n^* \cdot (M^{-1}f^{(n)}(x))$  maps the Bloch eigenfunctions of  $L_n$  to the eigenfunctions of  $L$ .

To complete the proof, it is sufficient to prove decompositions theorem for smooth **complex** finite-gap operators.

### 1.3 Decomposition for smooth complex periodic potentials

Consider a hyperelliptic Riemann surface  $\Gamma$  with divisor  $\mathcal{D}$ . We assume, that the corresponding potential  $u(x)$  is regular and periodic with the period  $T$ , but may be complex.

Near the point infinity it is natural to use the local coordinate  $1/k$  where  $k = p(\gamma)$ ,  $\lambda = k^2 + O(1)$ . In the neighbourhood of infinity we have:

$$\Psi(x, \gamma) = e^{ikx} \left[ 1 + \frac{\phi_1(x)}{k} + O\left(\frac{1}{k^2}\right) \right], \quad \Psi^*(y, \gamma) = e^{-iky} \left[ 1 - \frac{\phi_1(y)}{k} + O\left(\frac{1}{k^2}\right) \right],$$

$$\phi_1(x + T) = \phi_1(x).$$

Let us denote

$$\Xi(x, y, \gamma) = \Psi(x, \gamma) \Psi^*(y, \gamma) \frac{(\lambda(\gamma) - \lambda_1) \dots (\lambda(\gamma) - \lambda_g)}{2\sqrt{(\lambda(\gamma) - E_0) \dots (\lambda(\gamma) - E_{2g})}} d\lambda(\gamma)$$

There exists a constant  $K_0$  such, that in the domain  $|k| > K_0$  the function  $1/k$  is a well-defined local coordinate and the function  $\Psi(x, \gamma)e^{-ikx}$  is holomorphic in  $1/k$  and  $x$  in the domain  $|k| > K_0$ ,  $|\operatorname{Im} x| < \epsilon$ .

For sufficiently large  $k$  we have

$$\Xi(x, y, \gamma) = e^{ik(x-y)} \left[ 1 + \frac{\phi_1(x) - \phi_1(y)}{k} + \frac{\chi_2(k, x, y)}{k^2} \right] dk$$

where  $\chi_2(x, y, k)$  is holomorphic in  $1/k$ ,  $x, y$  in the domain  $|k| > K_0$ ,  $|\operatorname{Im} x| < \epsilon$ ,  $|\operatorname{Im} y| < \epsilon$ .

Our purpose is to study the convergence of the Fourier transformation. Let  $f(x)$  be either a Schwartz class function or a Bloch-periodic function.

1. Case 1. The integral Fourier transform. Let  $f(x)$  be a smooth finite function.

Let us define:

$$\hat{f}(\gamma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi^*(y, \gamma) f(y) dy$$

$$(\hat{\mathcal{S}}(K)f)(x) = \oint_{\substack{\text{Im } p(\gamma)=0, \\ |\text{Re } p(\gamma)| \leq K}} \hat{f}(\gamma) \Psi(x, \gamma) \frac{(\lambda(\gamma) - \lambda_1) \dots (\lambda(\gamma) - \lambda_g)}{2\sqrt{(\lambda(\gamma) - E_0) \dots (\lambda(\gamma) - E_{2g})}} d\lambda(\gamma) \quad (13)$$

Our purpose is to show, that  $(\hat{\mathcal{S}}(K)f)(x)$  converges to  $f(x)$  as  $K \rightarrow \infty$ . It is easy to see, that

$$(\hat{\mathcal{S}}(K)f)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(K, x, y) f(y) dy$$

where

$$S(K; x, y) = \oint_{\substack{\text{Im } p(\gamma)=0, \\ |\text{Re } p(\gamma)| \leq K}} \Xi(x, y, \gamma) \quad (14)$$

In this section we shall prove the following theorem:

**Propositions 1.** *Assume, that:*

- (a) *The spectral curve  $\Gamma$  is regular (has no multiple points),*
- (b) *The contour  $\text{Im } p(\gamma) = 0$  is regular, i.e.  $dp(\gamma) \neq 0$  everywhere at this contour.*

*Then:*

- (a) *The kernel  $S(K, x, y)$  has the following structure*

$$S(K, x, y) = S_{\text{classical}}(K, x, y) + S_{\text{correction}}(K, x, y)$$

*where*

$$S_{\text{classical}}(K, x, y) = \frac{2 \sin(K(x - y))}{x - y}$$

*is the corresponding kernel for the “standard” integral Fourier transform, and  $S_{\text{correction}}(K, x, y)$  uniformly converges at any compact set in the  $(x, y)$ -plane to a continuous function  $S_{\text{correction}}(\infty, x, y)$ .*

- (b) *Let  $x$  does not belong to the support of  $f(y)$ . Then  $(\hat{\mathcal{S}}(K)f)(x) \rightarrow 0$  for  $K \rightarrow \infty$ , and  $S_{\text{correction}}(\infty, x, y) \equiv 0$ . Moreover  $(\hat{\mathcal{S}}(K)f)(x) \rightarrow 0$  faster than any degree of  $K$ .*



2. Case 2. The discrete Fourier transform. Let  $f(x)$  be Bloch-periodic with the period  $T$ :

$$f(x + T) = \varkappa_0 f(x),$$

where  $\varkappa_0 = e^{iT\varphi_0}$  is an unitary multiplier  $|\varkappa_0| = 1$ . Consider the set of all points  $\kappa_j$  such, that  $e^{iT p(\kappa_j)} = \varkappa_0$ . Let us define

$$\hat{f}(\kappa_j) = \frac{1}{T} \int_0^T \Psi^*(\kappa_j, y) f(y) dy$$

The multipliers in the integrand have opposite Bloch multipliers, therefore we can integrate over any basic period. Let us define

$$(\hat{\mathcal{S}}(N)f)(x) = \sum_{|(p(\kappa_j) - \varphi_0)T| \leq 2\pi N} \hat{f}(\kappa_j) \Psi(\kappa_j, x) \frac{(\lambda(\kappa_j) - \lambda_1) \dots (\lambda(\kappa_j) - \lambda_g)}{2\sqrt{(\lambda(\kappa_j) - E_0) \dots (\lambda(\kappa_j) - E_{2g})}} \left[ \frac{d\lambda(\gamma)}{dp(\gamma)} \right] \Big|_{\lambda=\kappa_j}$$

We have

$$(\hat{\mathcal{S}}(N)f)(x) = \int_0^T S(N, x, y) f(y) dy$$

where

$$S(N, x, y) = \frac{1}{T} \sum_{|(p(\kappa_j) - \varphi_0)T| \leq 2\pi N} \frac{\Xi(\kappa_j, x, y)}{dp(\kappa_j)} \quad (15)$$

**Propositions 2.** Assume, that all points  $\kappa_j$  such, that  $e^{iT p(\kappa_j)} = \varkappa_0$  are regular:

- (a) They do not coincide with the multiple points (if they exists).
- (b)  $dp(\kappa_j) \neq 0$  for all  $j$ .

Then

- (a) The kernel  $S(N, x, y)$  has the following structure

$$S(N, x, y) = S_{\text{classical}}(N, x, y) + S_{\text{correction}}(N, x, y)$$

where

$$S_{\text{classical}}(N, x, y) = \frac{e^{i\phi_0(x-y)} \sin\left(\frac{\pi(2N+1)}{T}(x-y)\right)}{T \sin\left(\frac{\pi}{T}(x-y)\right)}$$

is the corresponding kernel for the “standard” discrete Fourier transform, and  $S_{\text{correction}}(N, x, y)$  uniformly converges in the  $(x, y)$ -plane to a continuous function  $S_{\text{correction}}(\infty, x, y)$ .

- (b) Let a point  $x$  does not belong to the support of  $f(y)$ . Then  $(\hat{\mathcal{S}}(N)f)(x) \rightarrow 0$  for  $N \rightarrow \infty$ , and  $S_{\text{correction}}(\infty, x, y) \equiv 0$ .

Let us prove the first part of Proposition 1.

Let  $S(K, x, y)$  be the kernel defined by formula (14)

We assume, that the orientation on this contour is defined by  $\text{Re } dp(\gamma) > 0$ .

Let us fix a sufficiently large constant  $K_0$ . Then we can write

$$S(K; x, y) = I_1(x, y) + I_2(K, x, y) + I_3(K, x, y) + I_4(K, x, y)$$

$$I_1(x, y) = \oint_{\substack{\text{Im } p(\gamma)=0, \\ |\text{Re } p(\gamma)| \leq K_0}} \Xi(x, y, \gamma)$$

$$I_2(K, x, y) = \left[ \int_{-K}^{-K_0} + \int_{K_0}^K \right] e^{ik(x-y)} dk$$

$$I_3(K, x, y) = \left[ \int_{-K}^{-K_0} + \int_{K_0}^K \right] e^{ik(x-y)} \left[ \frac{\phi_1(x) - \phi_1(y)}{k} \right] dk$$

$$I_4(K, x, y) = \left[ \int_{-K}^{-K_0} + \int_{K_0}^K \right] e^{ik(x-y)} \frac{\chi_2(k, x, y)}{k^2} dk$$

A standard calculation implies:

$$I_2(K, x, y) = \frac{2 \sin(K(x-y))}{x-y} - \frac{2 \sin(K_0(x-y))}{x-y}$$

Let us denote:

$$S_{\text{classical}}(K, x, y) = I_2(K, x, y) + \frac{2 \sin(K_0(x-y))}{x-y}$$

$$S_{\text{correction}}(K, x, y) = I_1(x, y) + I_3(K, x, y) + I_4(K, x, y) - \frac{2 \sin(K_0(x-y))}{x-y}$$

The functions  $I_1(x, y)$  and  $-\frac{2\sin(K_0(x-y))}{x-y}$  does not depend on  $K$  and are continuous in both variables. Integral  $I_4(K, x, y)$  absolutely converges as  $K \rightarrow \infty$ , therefore the limiting function is continuous in  $x, y$ . We also have

$$I_3(K, x, y) = \frac{2}{i} \text{Si}(K(x-y))(\phi_1(x) - \phi_1(y)) - \frac{2}{i} \text{Si}(K_0(x-y))(\phi_1(x) - \phi_1(y)),$$

where

$$\text{Si}(x) = \int_0^x \frac{\sin(t)}{t} dt,$$

therefore it uniformly converges to the continuous function

$$I_3(\infty, x, y) = \frac{\pi}{i} \text{sign}(x-y)(\phi_1(x) - \phi_1(y)) - \frac{2}{i} \text{Si}(K_0(x-y))(\phi_1(x) - \phi_1(y)),$$

This completes the proof.  $\square$

**Corollary 2.** *The kernel  $S(\infty, x, y) = \lim_{K \rightarrow \infty} S(K, x, y)$  is well-defined distribution and*

$$S(\infty, x, y) = 2\pi\delta(x-y) + S_{\text{correction}}(\infty, x, y)$$

Let us prove the first part of Proposition 2.

Let us remark, that the proof of Propositions 2 completely analogous to the proof of decomposition theorem for real regular non-stationary Shrödinger operators, suggested in [4].

Consider sufficiently large  $N_0$ . It is natural to write

$$S(N, x, y) = I_1(x, y) + I_2(N, x, y) + I_3(N, x, y) + I_4(N, x, y)$$

where

$$\begin{aligned} I_1(x, y) &= \frac{1}{T} \sum_{|(p(\kappa_j) - \varphi_0)T| \leq 2\pi N_0} \frac{\Xi(\kappa_j, x, y)}{dp(\kappa_j)}, \\ I_2(N, x, y) &= \frac{1}{T} \left[ \sum_{j=-N}^{-1-N_0} + \sum_{j=N_0+1}^N \right] e^{\left(\frac{2\pi i}{T}N + i\varphi_0\right)(x-y)} \\ I_3(N, x, y) &= \frac{1}{T} \left[ \sum_{j=-N}^{-1-N_0} + \sum_{j=N_0+1}^N \right] \frac{e^{\left(\frac{2\pi i}{T}j + i\varphi_0\right)(x-y)}}{\frac{2\pi}{T}j + \varphi_0} (\phi_1(x) - \phi_1(y)) \end{aligned}$$

$$I_4(N, x, y) = \frac{1}{T} \left[ \sum_{j=-N}^{-1-N_0} + \sum_{j=N_0+1}^N \right] e^{(\frac{2\pi i}{T}j + i\varphi_0)(x-y)} \frac{\chi_2(2\pi Tj + \varphi_0, x, y)}{(\frac{2\pi}{T}j + \varphi_0)^2}$$

A standard calculation implies:

$$I_2(N, x, y) = \frac{e^{i\varphi_0(x-y)}}{T} \frac{\sin\left(\frac{\pi(2N+1)x}{T}\right)}{\sin\left(\frac{\pi x}{T}\right)} - \frac{e^{i\varphi_0(x-y)}}{T} \frac{\sin\left(\frac{\pi(2N_0+1)x}{T}\right)}{\sin\left(\frac{\pi x}{T}\right)}$$

Let us denote:

$$S_{\text{classical}}(N, x, y) = I_2(N, x, y) + \frac{e^{i\varphi_0(x-y)}}{T} \frac{\sin\left(\frac{\pi(2N_0+1)x}{T}\right)}{\sin\left(\frac{\pi x}{T}\right)} = \frac{e^{i\varphi_0(x-y)}}{T} \frac{\sin\left(\frac{\pi(2N+1)x}{T}\right)}{\sin\left(\frac{\pi x}{T}\right)}$$

$$S_{\text{correction}}(N, x, y) = I_1(x, y) + I_3(N, x, y) + I_4(N, x, y) - \frac{e^{i\varphi_0(x-y)}}{T} \frac{\sin\left(\frac{\pi(2N_0+1)x}{T}\right)}{\sin\left(\frac{\pi x}{T}\right)}$$

The term  $I_1(x, y)$  is continuous in  $x, y$ ,  $I_4(N, x, y)$  uniformly converges to a continuous function.

The term  $I_3(N, x, y)$  requires some extra attention. It can be written as:

$$I_3(N, x, y) = \frac{1}{2\pi} e^{i\varphi_0(x-y)} \left[ \sum_{j=-N}^{1-N_0} + \sum_{j=N_0+1}^N \right] \frac{e^{(\frac{2\pi i}{T}j)(x-y)}}{j} (\phi_1(x) - \phi_1(y)) + \\ - \frac{\varphi_0}{T} e^{i\varphi_0(x-y)} \left[ \sum_{j=-N}^{1-N_0} + \sum_{j=N_0+1}^N \right] \frac{e^{(\frac{2\pi i}{T}j)(x-y)}}{(\frac{2\pi}{T}j + \varphi_0)(\frac{2\pi}{T}j)} (\phi_1(x) - \phi_1(y))$$

The second term uniformly converges to a continuous function in  $x, y$ .

Let us denote

$$S_1(N, z) = \left[ \sum_{j=-N}^1 + \sum_{j=1}^N \right] \frac{e^{ikx}}{k} = i \int_{-\pi}^z \left[ \frac{\sin\left(\frac{\pi(2N+1)}{T}(w)\right)}{\sin\left(\frac{\pi}{T}(w)\right)} - 1 \right] dw$$

Function  $S_1(N, z)$  is periodic with period  $2\pi$  and converges to  $i(\pi \text{sign}(z) - z)$  at the interval  $[-\pi, \pi]$  uniformly outside any neighbourhood of the point  $z = 0$ . We have

$$I_3(N, x, y) = \frac{1}{2\pi} e^{i\varphi_0(x-y)} S_1(N, \frac{2\pi}{T}(x-y)) (\phi_1(x) - \phi_1(y)) + \text{regular terms}$$

therefore it also converges uniformly to a continuous kernel.  $\square$

**Corollary 3.** *The kernel  $S(\infty, x, y) = \lim_{N \rightarrow \infty} S(N, x, y)$  is well-defined distribution and*

$$S(\infty, x, y) = \sum_j \delta(x - y - jT) + S_{\text{correction}}(\infty, x, y)$$

To continue the proof we need the following:

**Lemma 6.** *Let  $f(y)$  be a smooth function with compact support.*

$$\hat{f}(\gamma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi^*(y, \gamma) f(y) dy$$

*Then for any  $n$  there exist a constant  $C_n = C_n(u[y], f[y])$  such, that for sufficiently large  $\lambda(\gamma)$  we have*

$$|\hat{f}(\gamma)| \leq \frac{C_n}{|\lambda(\gamma)|^n} \max_{y \in \text{supp } f(y)} |e^{-ip(\gamma)y}|$$

**Proof.** By definition, we have  $L^n \Psi^*(x, \gamma) = \lambda(\gamma)^n \Psi^*(x, \gamma)$ . Therefore

$$\lambda(\gamma)^n \int_{-\infty}^{\infty} \Psi^*(y, \gamma) f(y) dy = \int_{-\infty}^{\infty} [L^n \Psi^*(y, \gamma)] f(y) dy.$$

Function  $f(y)$  has finite support, therefore we can eliminate all derivatives of  $\Psi^*(y, \gamma)$  integrating by parts, and we obtain

$$\begin{aligned} & \lambda(\gamma)^n \int_{-\infty}^{\infty} \Psi^*(y, \gamma) f(y) dy = \\ &= \int_{-\infty}^{\infty} \Psi^*(y, \gamma) P_n(f(y), f'(y), \dots, f^{(2n)}(y), u(y), u'(y), \dots, u^{(2n-2)}(y)) dy. \end{aligned}$$

where  $P(\dots)$  is a polynomial. The function  $e^{ip(\gamma)y} \Psi^*(y, \gamma)$  is uniformly bounded for all  $y$  and sufficiently large  $\lambda(\gamma)$ , therefore we obtain the desired estimate.

Let us prove the second part of Proposition 1.

Consider the integral representation (13) for  $(\hat{\mathcal{S}}(K)f)(x)$ . The integrand is holomorphic in  $\gamma$  at the finite part of  $\Gamma$ . From Lemma 6 we see that

integral (13) absolutely converges as  $K \rightarrow \infty$  in the upper half-plane if  $x > \text{supp } f(y)$  or in the lower half-plane if  $x < \text{supp } f(y)$ . Moreover the integrand exponentially decreases in the corresponding half-plane, therefore it is equal to 0. The integrand decays at infinity faster then any degree of  $K$ , therefore the integral fast decays as  $K \rightarrow \infty$ .

Let us finish the proof of the second part of Proposition 2. We shall use the following integral representation for  $S(N, x, y)$

$$S(N, x, y) = \frac{1}{2\pi} \oint_{\beta_N} \frac{\Xi(\kappa_j, x, y)}{e^{ip(\kappa_j)T} - \varkappa_0}$$

where  $\beta_N$  denotes the following contour (we assume  $N$  to be sufficiently large and  $p(\gamma)$  is fixed near infinity as a single-valued function):

$$\beta_N = \beta_N^{(1)} \cup \beta_N^{(2)} \cup \beta_N^{(3)} \cup \beta_N^{(4)}$$

$$\beta_N^{(1)} = \{\text{Im } p(\gamma) = -N, |(\text{Re } p(\kappa_j) - \varphi_0)T| \leq 2\pi(N + 1/2)\},$$

$$\beta_N^{(2)} = \{|\text{Im } p(\gamma)| \leq N, (\text{Re } p(\kappa_j) - \varphi_0)T = 2\pi(N + 1/2)\},$$

$$\beta_N^{(3)} = \{\text{Im } p(\gamma) = N, |(\text{Re } p(\kappa_j) - \varphi_0)T| \leq 2\pi N\},$$

$$\beta_N^{(4)} = \{|\text{Im } p(\gamma)| \leq N, (\text{Re } p(\kappa_j) - \varphi_0)T = -2\pi(N + 1/2)\},$$

We choose the orientation on  $\beta_N$  by assuming, that infinite point lies outside the contour.

By calculating the residues we immediately obtain formula (15).

**Remark 2.** *At all multiple points (if they exist) we have  $\text{Im } p(E_j) = 0$ , therefore all of them lie inside the contour  $\beta_N$ . If we have a holomorphic differential on a singular curve, the sum of residues at singular points is equal to zero, therefore they do not affect our integral.*

We have

$$(\hat{S}(N)f)(x) = \frac{1}{2\pi} \oint_{\beta_N} \int_{x_0}^{x_0+T} \frac{\Xi(\kappa_j, x, y)}{e^{ip(\kappa_j)T} - \varkappa_0} f(y) dy$$

Let us choose  $x = x_0$ . Support  $f(y)$  does not contain  $x$ , therefore we can write

$$(\hat{\mathcal{S}}(N)f)(x) = \frac{1}{2\pi} \oint_{\beta_N} \int_{x+\varepsilon}^{x+T-\varepsilon} \frac{\Xi(\kappa_j, x, y)}{e^{ip(\kappa_j)T} - \varkappa_0} f(y) dy$$

for some  $\varepsilon > 0$ .

From Lemma 6 it follows, that for any  $M$  there exist constants  $D_M$  such, that

$$\left| \int_{x+\varepsilon}^{x+T-\varepsilon} \frac{\Xi(\kappa_j, x, y)}{e^{ip(\kappa_j)T} - \varkappa_0} f(y) dy \right| \leq D_M e^{-N\varepsilon T} \quad \text{on } \beta_1, \beta_3$$

$$\left| \int_{x+\varepsilon}^{x+T-\varepsilon} \frac{\Xi(\kappa_j, x, y)}{e^{ip(\kappa_j)T} - \varkappa_0} f(y) dy \right| \leq \frac{D_M}{N^M} \quad \text{on } \beta_2, \beta_4$$

Therefore  $\hat{\mathcal{S}}(N)f(x)$  tends to 0 faster, that any degree of  $N$  as  $N \rightarrow \infty$ .

## 1.4 The reconstruction formula for singular potentials

To complete the proof, let us check formula (7). Let us denote

$$\eta(\gamma) = \frac{\tilde{\Psi}(x, \gamma)}{\Psi(x, \gamma)}.$$

Then

$$\begin{aligned} \hat{f}(\gamma) &= \frac{1}{2\pi\eta(\sigma\gamma)} \int_{-\infty}^{\infty} \tilde{\Psi}(y, \sigma\gamma) f(y) dy = \\ &= \frac{1}{2\pi\eta(\sigma\gamma)} \int_{-\infty}^{\infty} \frac{(Q_1^* \cdot \dots \cdot Q_n^* \cdot Q_n \cdot \dots \cdot Q_1 \cdot \tilde{\Psi}(y, \sigma\gamma)) f(y)}{(\lambda(\gamma) - l_1) \dots (\lambda(\gamma) - l_n)} dy = \\ &= \frac{1}{2\pi\eta(\sigma\gamma)} \int_{-\infty}^{\infty} \frac{(Q_n \cdot \dots \cdot Q_1 \cdot \tilde{\Psi}(y, \sigma\gamma)) (Q_n \cdot \dots \cdot Q_1 \cdot f(y))}{(\lambda(\gamma) - l_1) \dots (\lambda(\gamma) - l_n)} dy = \\ &= \frac{1}{2\pi\eta(\sigma\gamma)} \int_{-\infty}^{\infty} \tilde{\Psi}^{(n)}(y, \sigma\gamma) f^{(n)}(y) dy = \frac{1}{\eta(\sigma\gamma)} \hat{f}^{(n)}(\gamma). \end{aligned}$$

We have

$$f^{(n)}(x) = \oint_{\operatorname{Im} p(\gamma)=0} \hat{f}^{(n)}(\gamma) \tilde{\Psi}^{(n)}(x, \gamma) \frac{(\lambda - \lambda_1(x_0)) \dots (\lambda - \lambda_{g-n}(x_0)) (\lambda - l_1) \dots (\lambda - l_n)}{2\sqrt{(\lambda - E_0) \dots (\lambda - E_{2g})}} d\lambda,$$

where  $\lambda = \lambda(\gamma)$ .

$$\begin{aligned} f(x) &= Q_1^* \cdot \dots \cdot Q_n^* \cdot (M^{-1} f^{(n)}(x)) = \\ &= \oint_{\operatorname{Im} p(\gamma)=0} \hat{f}^{(n)}(\gamma) \tilde{\Psi}(x, \gamma) \frac{(\lambda - \lambda_1(x_0)) \dots (\lambda - \lambda_{g-n}(x_0))}{2\sqrt{(\lambda - E_0) \dots (\lambda - E_{2g})}} d\lambda = \\ &= \oint_{\operatorname{Im} p(\gamma)=0} \eta(\sigma\gamma) \eta(\gamma) \hat{f}(\gamma) \Psi(x, \gamma) \frac{(\lambda - \lambda_1(x_0)) \dots (\lambda - \lambda_{g-n}(x_0))}{2\sqrt{(\lambda - E_0) \dots (\lambda - E_{2g})}} d\lambda = \end{aligned}$$

Taking into account, that

$$\eta(\sigma\gamma) \eta(\gamma) = \frac{(\lambda(\gamma) - \lambda_1) \dots (\lambda(\gamma) - \lambda_g)}{(\lambda - \lambda_1(x_0)) \dots (\lambda - \lambda_{g-n}(x_0))},$$

we obtain formula (7).

## 2 Appendix. Proof of Statement 3

In this Appendix we present a proof of the Statement 3.

To be precise, we prove the following:

**Theorem.** *Assume, that we have the space  $\mathcal{F}_X$  associated with a either a decaying at infinity potential with  $N$  singular points of orders  $n_1, \dots, n_N$  or a periodic potentials with  $N$  singular points of orders  $n_1, \dots, n_N$  at the period. In the periodic case we assume that an unitary Bloch multiplier  $\varkappa_0$ ,  $|\varkappa_0| = 1$  is fixed.*

1. Denote by  $l_X$  the following number  $l_X = \left[\frac{n_1+1}{2}\right] + \left[\frac{n_2+1}{2}\right] + \dots + \left[\frac{n_N+1}{2}\right]$ , where  $[ \ ]$  is the integer part of a number.

*The exists an  $l_X$ -dimensional subspace of  $\mathcal{F}_X$  such that our scalar product is negative defined on it.*



2. Any subspace of dimension  $d > l_x$  has non-zero intersection with  $\mathcal{F}_X^0$ , i.e. contains at least one function with positive square.

The proof the second part is straightforward. A function from the space  $\mathcal{F}_X$  lies in  $\mathcal{F}_X^0$  if it satisfies exactly  $l_x$  linear equations: all singular terms in the expansions near points  $x_j$  are equal to 0. We have  $d$ -dimensional subspace with  $d > l_x$ , therefore this system of equations has a least one non-trivial solution.

To prove the first part of the Theorem, let us construct these negative subspaces explicitly. It is convenient to consider the decaying and the periodic cases separately.

## 2.1 Decaying at infinity case.

To start with, let us prove three technical lemmas.

**Lemma 7.** Assume, that we have only one singular point  $x_1 = 0$  of order  $n$ . For any  $n$  the functions  $1/x^{n-2l}$ ,  $l = 0, 1, \dots, \lfloor \frac{n-1}{2} \rfloor$  generate a zero subspace with respect to our scalar product:

$$\left\langle \frac{1}{x^{n-2k}}, \frac{1}{x^{n-2l}} \right\rangle = \int_{-\infty}^{+\infty} \frac{dx}{x^{2n-2k-2l}} = 0.$$

Here we use our standard rule that the integration contour goes around zero in the complex domain,  $2k < n$ ,  $2l < n$ .

The proof is evident.

**Lemma 8.** Assume, that we have only one singular point at the point  $x_1 = 0$  of order  $n$ ,  $\mathcal{N}$  is an integer such, that  $\mathcal{N} > n$ .

Consider the following collection of functions  $\Xi_l(x, \varepsilon)$ ,  $l = 0, 1, \dots, \lfloor \frac{n-1}{2} \rfloor$  in our space  $\mathcal{F}_X$ :

$$\Xi_l(x, \varepsilon) = \frac{1}{\sqrt{\varepsilon}} \left[ \frac{\varepsilon}{x} \right]^{n-2l} e^{-[x/\varepsilon]^{2\mathcal{N}}},$$

The Gram matrix  $g_{kl}^n$  for this collection of functions

$$g_{kl}^n = \langle \Xi_k(x, \varepsilon), \Xi_l(x, \varepsilon) \rangle \tag{16}$$

is negative defined and does not depend on  $\varepsilon$ .

**Proof.** Consider any linear combination of these functions

$$f(x) = \sum_{k=0}^{n-1} d_k \Xi_k(x, \varepsilon)$$

We have

$$\begin{aligned} \langle f, f \rangle &= \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} d_k \bar{d}_l \left[ \frac{\varepsilon}{x} \right]^{2n-2k-2l} [e^{-2[x/\varepsilon]^{2N}} - 1] dx + \\ &\quad + \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} d_k \bar{d}_l \left[ \frac{\varepsilon}{x} \right]^{2n-2k-2l} dx \end{aligned}$$

The second integral is equal to zero by Lemma 7 and the integrand in the first integral is real, regular and strictly negative, therefore

$$\langle f, f \rangle < 0.$$

The second statement immediately follows from the scaling properties.

**Lemma 9.** Assume, that we have only one singular point at the point  $x_1 = 0$  of order  $n$ ,  $N$  is an integer such, that  $N > n$ . Let us fix an interval  $[-L, L]$  where  $L$  is either a positive number or  $+\infty$ .

Consider the following collection of functions  $\Xi_l(x, \varepsilon)$ ,  $l = 0, 1, \dots, \left[\frac{n-1}{2}\right]$  in our space  $\mathcal{F}_X$ :

$$\Xi_l(x, \varepsilon) = \frac{1}{\sqrt{\varepsilon}} \left[ \frac{\varepsilon}{x} \right]^{n-2l} \cdot e^{-[x/\varepsilon]^{2N}} \cdot \zeta(x),$$

where  $\zeta(0) \neq 0$ ,  $\zeta(x)$  is bounded on the interval  $[-L, L]$  and smooth inside the interval,  $\zeta'(0) = \zeta''(0) = \dots = \zeta^{(2N-1)}(0) = 0$ . Define the Gram matrix  $\tilde{g}_{kl}^n(\varepsilon)$  for this collection of functions by

$$\tilde{g}_{kl}^n(\varepsilon) = \int_{-L}^{+L} \Xi_k(x, \varepsilon) \overline{\Xi_l(\bar{x}, \varepsilon)} dx. \quad (17)$$

Then

$$\tilde{g}_{kl}^n(\varepsilon) \rightarrow g_{kl}^n \cdot |\zeta^2(0)| \quad \text{as } \varepsilon \rightarrow 0,$$

where  $g_{kl}^n$  are the scalar products from Lemma 8. Here we use our standard rule, that the integration contour goes around the singular point  $x = 0$ .

**Proof.** Let us make the following substitution:  $x = \varepsilon y$ . We have

$$\begin{aligned}
\tilde{g}_{kl}^n(\varepsilon) &= \int_{-L/\varepsilon}^{+L/\varepsilon} \left[ \frac{1}{y} \right]^{2n-2k-2l} \cdot e^{-2y^{2\mathcal{N}}} \cdot \zeta(\varepsilon y) \overline{\zeta(\varepsilon \bar{y})} dy = \\
&= \zeta(0) \overline{\zeta(0)} \int_{-L/\varepsilon}^{+L/\varepsilon} \left[ \frac{1}{y} \right]^{2n-2k-2l} \cdot e^{-2y^{2\mathcal{N}}} dy + \\
&+ \int_{-L/\varepsilon}^{+L/\varepsilon} \left[ \frac{1}{y} \right]^{2n-2k-2l} \cdot \left[ \zeta(\varepsilon y) \overline{\zeta(\varepsilon \bar{y})} - \zeta(0) \overline{\zeta(0)} \right] \cdot e^{-2y^{2\mathcal{N}}} dy
\end{aligned}$$

The first integral converges to  $g_{kl}^n |\zeta^2(0)|$  as  $\varepsilon \rightarrow 0$ . The preexponent in the second integral is bounded and uniformly converges to 0 at any compact interval, therefore this integral converges to 0. The proof is finished.

Consider now the generic decaying at infinity case. We assume that our potential has  $N$  singular points  $x_1, \dots, x_N$  with the multiplicities  $n_1, \dots, n_N$ . Let  $\mathcal{N} = \max(n_1, \dots, n_N) + 1$ .

Consider the following collection of functions

$$\Xi_{lj}(x, \varepsilon) = \frac{1}{\sqrt{\varepsilon}} \cdot \left[ \frac{\varepsilon}{x - x_j} \right]^{n_j - 2l} e^{-\left[ \frac{x - x_j}{\varepsilon} \right]^{2\mathcal{N}}} \cdot \zeta_j(x),$$

where

$$\zeta_j(x) = \left[ \prod_{\substack{k \neq j \\ k=1, \dots, N}} \frac{((x - x_j)^{2\mathcal{N}} - (x_k - x_j)^{2\mathcal{N}})^2}{((x - x_j)^{2\mathcal{N}} - (x_k - x_j)^{2\mathcal{N}})^2 + 1} \right]^{\mathcal{N}}$$

$$l = 0, \dots, \left\lfloor \frac{n_j - 1}{2} \right\rfloor, j = 1, \dots, N.$$

**Lemma 10.** *All functions  $\Xi_{lj}$  belong to the space  $\mathcal{F}_X$ .*

**Proof.** The function  $\Xi_{lj}(x, \varepsilon)$  are symmetric in  $(x - x_j)$  if  $n_j$  is even or skew-symmetric in  $(x - x_j)$  if  $n_j$  is odd. At all other singular points  $x_l$  they have zeroes of order  $2\mathcal{N} \geq n_l + 1$ . At infinity they decay exponentially, therefore all conditions are fulfilled.

**Lemma 11.** *The scalar products of the functions defined above have the following form*

$$\langle \Xi_{l_1 j_1}(x, \varepsilon) \Xi_{l_2 j_2}(x, \varepsilon) \rangle = g_{l_1 l_2}^{n_{j_1}} \cdot \zeta_{j_1}^2(0) \cdot \delta_{j_1 j_2} + O(1) \quad \text{as } \varepsilon \rightarrow 0. \quad (18)$$

where  $g_{kl}^n$  denotes the Gram matrix defined by (16).

**Proof.** Let  $j_1 \neq j_2$ . Assume, that  $x_{j_1} < x_{j_2}$ ,  $2L = x_{j_2} - x_{j_1}$ . The product  $\Xi_{l_1 j_1}(x, \varepsilon) \Xi_{l_2 j_2}(x, \varepsilon)$  is regular in the whole  $x$ -line. Consider the following system of intervals in the  $x$ -line.

$$\mathcal{I}_1 = ]-\infty, x_{j_1} - L], \quad \mathcal{I}_2 = [x_{j_1} - L, x_{j_1} + L], \quad \mathcal{I}_3 = [x_{j_1} + L, x_{j_2} + L], \quad \mathcal{I}_4 = [x_{j_2} + L, +\infty[.$$

Then we have the following estimates:

$$|\Xi_{l_1 j_1}(x, \varepsilon) \Xi_{l_2 j_2}(x, \varepsilon)| \leq \frac{\varepsilon^{n_{j_1} + n_{j_2} - 2l_1 - 2l_2 - 1}}{L^{n_{j_1} + n_{j_2} - 2l_1 - 2l_2}} e^{-2[L/\varepsilon]^{2N}} \quad \text{for } x \in \mathcal{I}_1 \cup \mathcal{I}_4.$$

$$|\Xi_{l_1 j_1}(x, \varepsilon) \Xi_{l_2 j_2}(x, \varepsilon)| \leq \frac{\varepsilon^{n_{j_1} + n_{j_2} - 2l_1 - 2l_2 - 1} [2N]^{2N} 3^{4N^2 - 2N}}{L^{n_{j_1} - 2l_1 - 4N^2}} e^{-[L/\varepsilon]^{2N}} \quad \text{for } x \in \mathcal{I}_2$$

$$|\Xi_{l_1 j_1}(x, \varepsilon) \Xi_{l_2 j_2}(x, \varepsilon)| \leq \frac{\varepsilon^{n_{j_1} + n_{j_2} - 2l_1 - 2l_2 - 1} [2N]^{2N} 3^{4N^2 - 2N}}{L^{n_{j_2} - 2l_2 - 4N^2}} e^{-[L/\varepsilon]^{2N}} \quad \text{for } x \in \mathcal{I}_3$$

From these estimates it follows that all integrals exponentially decays as  $\varepsilon \rightarrow 0$ .

For  $j_1 = j_2$  the asymptotic formula for the scalar products immediately follows from Lemma 9.

As a corollary of Lemma 11 we immediately obtain, that for sufficiently small  $\varepsilon > 0$  the scalar product on our system of functions is negative defined. It completes the proof for fast decaying case.

## 2.2 Periodic case.

To simplify the formulas we shall assume in this section that the period of the potential is equal to  $\pi$ .

Consider the following collection of functions

$$\Xi_{lj}(x, \varepsilon) = \frac{1}{\sqrt{\varepsilon}} \cdot \left[ \frac{\varepsilon}{\sin(x - x_j)} \right]^{n_j - 2l} e^{-\left[ \frac{\sin(x - x_j)}{\varepsilon} \right]^{2N}} \cdot \zeta_j(x, \varepsilon) \cdot e^{ic_j \alpha(x)},$$

where

$$\zeta_j(x, \varepsilon) = \left[ \prod_{\substack{k \neq j \\ k=1, \dots, N}} \frac{([\sin(x - x_j)]^{2N} - [\sin(x_k - x_j)]^{2N})^2}{([\sin(x - x_j)]^{2N} - [\sin(x_k - x_j)]^{2N})^2 + 1} \right]^N,$$

$$\alpha(x) = \frac{\int_0^x \prod_{k=1}^N [\sin(y - x_k)]^{2N} dy}{\int_0^\pi \prod_{k=1}^N [\sin(y - x_k)]^{2N} dy}$$

$l = 0, \dots, \left\lfloor \frac{n_j-1}{2} \right\rfloor$ ,  $j = 1, \dots, N$ , the constants  $c_j$  are chosen to provide the proper periodicity:

$$e^{ic_j} = (-1)^{n_j} \varkappa_0.$$

It is easy to check, that all functions  $\Xi_{lj}$  belong to the space  $\mathcal{F}_X$ .

**Lemma 12.** *The scalar products of the functions defined above have the following form*

$$\langle \Xi_{l_1 j_1}(x, \varepsilon) \Xi_{l_2 j_2}(x, \varepsilon) \rangle = g_{l_1 l_2}^{n_{j_1}} \cdot \zeta_{j_1}^2(0) \cdot \delta_{j_1 j_2} + O(1) \quad \text{as } \varepsilon \rightarrow 0. \quad (19)$$

where  $g_{kl}^n$  denotes the Gram matrix defined by (16).

**Proof.**

Let  $j_1 \neq j_2$ . Then all products  $\Xi_{l_1 j_1}(x, \varepsilon) \overline{\Xi_{l_2 j_2}(\bar{x}, \varepsilon)}$  are regular functions, uniformly exponentially converging to 0 as  $\varepsilon \rightarrow 0$ .

If  $j_1 = j_2$ , it is sufficient to introduce a new variable  $y = \sin(x)$  and to apply Lemma 9

This completes the proof of Statement 3.

### 3 Two remarks

**Remark 3.** *a) The main goal of our work is to develop the best analog of Fourier transform (with good multiplicative properties) on the spaces of functions defined on some special "canonical" contours in Riemann surfaces. It involves quite original spectral theory of singular operators on the whole*

*Real line.* This spectral theory is based on the special indefinite inner product of functions, including some singular functions. It is our main discovery. We fully realized this program for real singular finite-gap (i.e. algebro-geometrical) operators, but it certainly can be extended to the infinite-gap cases also. Darboux-Crum transformations play only technical role in some of our proofs, we do not consider them as a subject of our work.

b) In our work we invented following class of real potentials with the special isolated singularities: for all values of complex spectral parameter solutions  $\psi(x)$  should be locally meromorphic in the infinitesimal neighborhood of the corresponding singular points at the real  $x$ -axis. We found no classical or modern works where this property has been discussed. The analogous property has been used in [9] but in the whole complex  $x$ -plane (and for elliptic potentials only) as an assumption implying that this potential is algebro-geometrical – i.e. “singular finite gap”. In our opinion, this is the most essential idea in the work [9] comparable with our theory. No doubt, there are many other results in the series of articles of Gesztesy and Weikard, not related to our problems. But we do not discuss them.

c) There is a problem concerning completion of our functional classes. We proved the decomposition theorems for locally smooth (outside of singularities) functions but Hilbert Spaces do not work.

**Remark 4.** a) Deconinck and Segur claim in the paragraph 4.4 of their work [10] that according to the KdV dynamics (with hierarchy) the poles of finite-gap elliptic, rational and trigonometric solution can collapse to singular points by the triangular groups only containing  $n(n+1)/2$  items (which is true), entering and leaving this point in the complex  $x$ -plane along the directions of the vertices of the equilateral polygon (which is wrong for  $n > 2$ ). In fact, according to our results [1], [2] this point splits “approximately” (but not exactly) as a set of integer points in the equilateral triangle for all  $n = 1, 2, 3, \dots$ . Exactly  $[(n+1)/2]$  poles remain at the real axis which is a diagonal in this triangle. This number plays fundamental role in our results [1], [2] defining the contribution of each pole to the total number of negative squares in the functional space with indefinite metric where the singular Schrödinger operator is symmetric.

b) Until now we also cannot understand proof in the work [10] that their elliptic potentials closed to multisoliton limit are finite-gap. We will clarify this question later.

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